

University of California, Berkeley
Physics 105 Fall 2000 Section 2 (*Strovink*)

SOLUTION TO PROBLEM SET 12

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Reading:

105 Notes 14.6

Hand & Finch 10.1-10.2

1.

Consider a uniform cube of side L . Inside the cube is a scalar field ϕ that satisfies the wave equation with characteristic wavespeed c . At the surfaces of the cube, ϕ is required to vanish.

(a)

Show that for this system the total number of modes of vibration corresponding to frequencies between ν and $\nu + d\nu$ is $4\pi^2 L^3 \nu^2 d\nu / c^3$, if $\pi c/L \ll d\nu \ll \nu$.

Solution:

Let's first figure out what the normal modes look like. The wave equation inside the box is

$$\nabla^2 \phi - \frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2} = 0$$

You can solve this equation by separation of variables: Assume that the solution is of the form $\phi(x, y, z, t) = X(x)Y(y)Z(z)T(t)$, and you'll find that the solutions are of the form

$$\phi \propto e^{i(k_x x + k_y y + k_z z - \omega t)}, \text{ where } \omega^2 \equiv c^2(k_x^2 + k_y^2 + k_z^2).$$

In order to satisfy the boundary condition $\phi = 0$ on the edges of the box, the complex exponentials in x, y, z must all be sines, not cosines, and the numbers k_x, k_y, k_z must all be integer multiples of π/L . So the normal modes are

$$\phi \propto e^{i\omega t} \sin \frac{l\pi x}{L} \sin \frac{m\pi y}{L} \sin \frac{n\pi z}{L}$$

with l, m, n positive integers. The frequency of a given mode is $\nu = \omega/2\pi = (c/2\pi)|\vec{k}|$, where $\vec{k} = (k_x, k_y, k_z) = (\frac{l\pi}{L}, \frac{m\pi}{L}, \frac{n\pi}{L})$. We need to find the number of modes between ν and $\nu + d\nu$, but since ν and $|\vec{k}|$ are proportional, let's find the number between $|\vec{k}|$ and $|\vec{k}| + d|\vec{k}|$ instead.

Let $N(k)$ be the number of modes whose wave vector \vec{k} is of length less than k . If you picture the wave vectors as points in three-dimensional space, $N(k)$ is the number of points inside of one octant of a sphere of radius k . (It's only one octant because negative values of the integers l, m, n don't lead to physically distinct states.) Using the usual formula for the volume of the sphere, we get

$$N(k) = \frac{1}{8} \cdot \frac{4\pi}{3} k^3 \cdot \left(\begin{array}{c} \text{density of wave vectors} \\ \text{in } k\text{-space} \end{array} \right)$$

The last term is simply the number of allowed \vec{k} 's per unit volume in k -space. The allowed k -vectors are spaced on a cubic lattice of side π/L , so there is one vector per volume $(\pi/L)^3$. The density is therefore $(L/\pi)^3$ vectors per unit volume, and

$$N(k) = \frac{L^3 k^3}{6\pi^2}$$

Use the formula $k = 2\pi\nu/c$ to get $N(\nu) = 4\pi L^3 \nu^3 / 3c^3$ for the number of modes with frequency less than ν . The number of modes between ν and $\nu + d\nu$ is the differential of this:

$$dN = \frac{4\pi L^3}{c^3} \nu^2 d\nu$$

(b)

What would the result be for a (two-dimensional) square?

Solution:

In two dimensions, use the formula for the area of a one quadrant of a circle instead of the volume of one octant of a sphere, and use $(L/\pi)^2$ instead of $(L/\pi)^3$ for the density in k -space. Then

$$N(k) = \frac{1}{4} \pi k^2 \left(\frac{L}{\pi} \right)^2$$

Convert to frequency and differentiate to get

$$dN = \frac{2\pi L^2}{c^2} \nu d\nu$$

(c)

A (one-dimensional) rod?

Solution:

In one dimension, k -space is just a line, and so instead of the volume or the area, you just have the length. The same process gives

$$dN = \frac{2L}{c} d\nu$$

2. and 3. (double credit problem)

Consider a homogeneous isotropic *solid* medium, *i.e.* a medium that, unlike a liquid, is able to resist being twisted (it “supports a shear stress”). The Lagrangian density for such a medium is

$$\mathcal{L}' = \frac{1}{2}\rho \frac{\partial u_i}{\partial t} \frac{\partial u_i}{\partial t} - \frac{1}{2} \frac{\partial u_i}{\partial x_j} C_{ijkl} \frac{\partial u_k}{\partial x_l},$$

where summation over repeated indices is (definitely!) implied. In this expression, the field variables are $u_1(x_1, x_2, x_3, t)$, $u_2(x_1, x_2, x_3, t)$, and $u_3(x_1, x_2, x_3, t)$. These describe the (vector) displacement \mathbf{u} of a small element of the solid from its equilibrium position \mathbf{x} . (The *strain* is obtained by taking spatial derivatives of \mathbf{u} .) The mass density of the solid is ρ , which for small values of \mathbf{u} can be approximated as a constant. C_{ijkl} is the “fourth-rank tensor of elasticity”.

Exploiting the homogeneous medium’s isotropy, one can show that the most general form for C_{ijkl} is

$$C_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}),$$

where λ and μ , the so-called “Lamé constants”, determine all 81 of its elements. The inverse of the *compression modulus* λ is proportional to the compressibility of the medium, and the inverse of the *shear modulus* μ is proportional to the extent to which the medium can be twisted.

Notice that the Lagrangian density for a solid medium could in principle depend on 19 variables (3 field variables, 3×4 derivatives of 3 field

variables with respect to 4 independent variables, and 4 independent variables). In practice, our Lagrangian density has no dependence on the first and last category, so it is a function of only 12 variables.

Use the Euler-Lagrange equations for this Lagrangian density to derive the wave equations for compression waves ($\nabla \times \mathbf{u} = 0$) and for shear waves ($\nabla \cdot \mathbf{u} = 0$) in the solid. Obtain the phase velocity c for both cases, in terms of λ , μ , and ρ . Notice that an earthquake can propagate with more than one velocity!

Solution:

Start from the Euler-Lagrange equation

$$\frac{d}{dt} \left(\frac{\partial \mathcal{L}'}{\partial \dot{u}_n} \right) + \frac{d}{dx_m} \left(\frac{\partial \mathcal{L}'}{\partial \left(\frac{\partial u_n}{\partial x_m} \right)} \right) - \frac{\partial \mathcal{L}'}{\partial u_n} = 0$$

The first term just gives $\rho \ddot{u}_n$, and the third term is zero. Let’s figure out the second term.

$$\begin{aligned} \frac{\partial \mathcal{L}'}{\partial \left(\frac{\partial u_n}{\partial x_m} \right)} &= -\frac{1}{2} C_{nmkl} \frac{\partial u_k}{\partial x_l} - \frac{1}{2} C_{ijnm} \frac{\partial u_i}{\partial x_j} \\ &= -\frac{1}{2} (C_{nmij} + C_{ijnm}) \frac{\partial u_i}{\partial x_j} \\ &= -\lambda \delta_{nm} \frac{\partial u_i}{\partial x_i} - \mu \left(\frac{\partial u_n}{\partial x_m} + \frac{\partial u_m}{\partial x_n} \right). \end{aligned}$$

Taking the derivative with respect to x_m , we get

$$\begin{aligned} \frac{d}{dx_m} \left(\frac{\partial \mathcal{L}'}{\partial \left(\frac{\partial u_n}{\partial x_m} \right)} \right) &= \\ &= -\lambda \frac{\partial^2 u_i}{\partial x_n \partial x_i} - \mu \frac{\partial^2 u_n}{\partial x_m \partial x_m} - \mu \frac{\partial^2 u_m}{\partial x_m \partial x_n} \\ &= -(\lambda + \mu) (\nabla (\nabla \cdot \vec{u}))_n - \mu \nabla^2 u_n. \end{aligned}$$

So the full Euler-Lagrange equation is

$$\rho \frac{\partial^2 \vec{u}}{\partial t^2} - (\lambda + \mu) \nabla (\nabla \cdot \vec{u}) - \mu \nabla^2 \vec{u} = 0$$

Now let’s use this to get the wave equation for compression and shear waves. First, suppose $\nabla \times \vec{u} = 0$, so we have a compression wave. Then we can perform a trick to get rid of the

unwanted $\nabla(\nabla \cdot \vec{u})$ term in the wave equation: If $\nabla \times \vec{u} = 0$, then $\nabla \times (\nabla \times \vec{u}) = 0$. But there's a vector identity that says

$$\nabla \times (\nabla \times \vec{u}) = \nabla(\nabla \cdot \vec{u}) - \nabla^2 \vec{u}$$

so $\nabla(\nabla \cdot \vec{u}) = \nabla^2 \vec{u}$. Then the wave equation becomes

$$\rho \frac{\partial^2 \vec{u}}{\partial t^2} - (\lambda + 2\mu) \nabla^2 \vec{u} = 0$$

That's the wave equation for compression waves. The wave speed is given by $c_c^2 = (\lambda + 2\mu) / \rho$.

Shear waves are easier. Since $\nabla \cdot \vec{u} = 0$, the Euler-Lagrange equation becomes

$$\rho \frac{\partial^2 \vec{u}}{\partial t^2} - \mu \nabla^2 \vec{u} = 0$$

and the wave speed is given by $c_s^2 = \mu / \rho$.

4.

Consider an infinitely long continuous string in which the tension is τ . A mass M is attached to the string at $x = 0$. If a sinusoidal wave train with velocity ω/k is incident from the left, analyze the reflection and transmission that occur at $x = 0$. Define the reflection coefficient $R \equiv |\mathcal{R}|^2$ and the transmission coefficient $T \equiv |\mathcal{T}|^2$, where \mathcal{R} and \mathcal{T} are the reflected and transmitted amplitude ratios discussed in Lecture Notes section 14.6.

Show that R and T are given by $R = \sin^2 \theta$ and $T = \cos^2 \theta$, where $\tan \theta = M\omega^2 / 2k\tau$. [Hint: Consider carefully the boundary condition on the derivatives of the wave functions at $x = 0$.]

Solution:

Call the displacement y_1 for $x < 0$ and y_2 for $x > 0$. Then y_1 and y_2 are of the form

$$\begin{aligned} y_1(x, t) &= \text{Re}(Ae^{ikx-i\omega t} + Be^{-ikx-i\omega t}) \\ y_2(x, t) &= \text{Re}(Ce^{ikx-i\omega t}), \end{aligned}$$

where A, B, C are the (complex) amplitudes of the incident, reflected, and transmitted waves, respectively. The requirement that $y_1(0, t) = y_2(0, t)$ yields a real equation; we choose to solve

the complex equation of which that equation is the real part. It is

$$A + B = C.$$

If the point mass weren't there, we would impose the condition that the forces just to the left and right of $x = 0$ add up to zero. That requirement is not satisfied here, since the point mass at $x = 0$ is accelerating, and so the net force on it must be nonzero. Instead, we can apply Newton's second law: $F_{1y} + F_{2y} = M\ddot{y}$. Here F_{1y} means the y -component of the force exerted by the left half of the string on the mass. Clearly this is $-\tau \sin \phi$, where ϕ is the angle the left half of the string makes with the horizontal. The slope of the string is $dy_1/dx = \tan \phi$, but for small angles $\sin \phi \approx \tan \phi$, so we can say $F_{1y} = -\tau dy_1/dx$. Similarly, $F_{2y} = \tau dy_2/dx$. So our second boundary condition is

$$\tau \left(\frac{dy_2}{dx} - \frac{dy_1}{dx} \right) = M\ddot{y}$$

(The y on the right-hand-side can be either y_1 or y_2 , since they're equal at $x = 0$.) Using our expressions for y_1 and y_2 , we get

$$ik\tau(C - A + B) = -M\omega^2 C$$

Solving these two equations, we get

$$\begin{aligned} B &= \frac{-M\omega^2 A}{M\omega^2 + 2ik\tau} \\ C &= \frac{2ik\tau A}{M\omega^2 + 2ik\tau} \end{aligned}$$

The reflection and transmission coefficients are

$$\begin{aligned} R &= \left| \frac{B}{A} \right|^2 = \frac{M^2\omega^4}{M^2\omega^4 + 4k^2\tau^2} \\ T &= \left| \frac{C}{A} \right|^2 = \frac{4k^2\tau^2}{M^2\omega^4 + 4k^2\tau^2} \end{aligned}$$

If we define $T = \cos^2 \theta$ and $R = \sin^2 \theta$, then $\tan \theta = \sqrt{\frac{R}{T}} = \frac{M\omega^2}{2k\tau}$.

5.

Hand & Finch, Problem 10.1 (stability in a central force)

Solution:

(a)

From Hand and Finch equation 4.41, we have

$$\mu \ddot{r} = \frac{l^2}{\mu r^3} + \frac{n\beta}{r^{n+1}}$$

To find the radius r_o of a stationary circular orbit, we set $\ddot{r} = 0$.

$$0 = \frac{l^2}{\mu r_o^3} + \frac{n\beta}{r_o^{n+1}}$$

$$r_o = \left(-\frac{n\beta\mu}{l^2} \right)^{\frac{1}{n-2}}$$

In order for this quantity to exist, we must have $n\beta < 0$ and $n \neq 2$. (If $n = 2$, we can have (not stable) circular orbits at any r_o only if $\beta = -l^2/2\mu$.)

(b)

Let $r = r_o + \delta r(t)$, where $\delta r \ll r_o$. Plugging this into the above equation yields

$$\mu \ddot{\delta r} = \frac{l^2}{\mu (r_o + \delta r)^3} + \frac{n\beta}{(r_o + \delta r)^{n+1}} .$$

Now Taylor expand around $\delta r = 0$ to get

$$\mu \ddot{\delta r} = - \left(\frac{3l^2}{\mu r_o^4} + \frac{(n+1)n\beta}{r_o^{n+2}} \right) \delta r$$

$$\ddot{\delta r} = - \frac{l^2}{\mu} \left(-\frac{n\beta\mu}{l^2} \right)^{\frac{4}{2-n}} (2-n) \delta r$$

In order for the circular orbit to be stable, we must have the coefficient of δr in the above expression be < 0 . Since we already know that $n\beta < 0$, this requires that $n < 2$.

6.

Hand & Finch, p. 397, *Question 6* (upside-down pendulum)

Solution:

With our origin at the point of support of the pendulum, we have $x = l \sin \theta$ and $y = Y(t) - l \cos \theta$. Since we are interested in the case where θ is near π , we make the substitution $\theta = \pi + \psi$. Differentiating with respect to time yields

$$\dot{x} = -l\dot{\psi} \cos \psi$$

$$\dot{y} = \dot{Y} - l\dot{\psi} \sin \psi .$$

And so the kinetic and potential energies are:

$$T = \frac{m}{2}(\dot{x}^2 + \dot{y}^2)$$

$$= \frac{m}{2} \left(l^2 \dot{\psi}^2 - 2l\dot{Y}\dot{\psi} \sin \psi + \dot{Y}^2 \right)$$

$$U = mgy = m\omega_o^2 l (Y + l \cos \psi)$$

where $\omega_o = \sqrt{\frac{g}{l}}$. Applying the Euler-Lagrange equation yields

$$l\ddot{\psi} - \left(\ddot{Y} + \omega_o^2 l \right) \sin \psi = 0 .$$

Now, following Hand & Finch, we plug in $Y(t) = Y_o \cos \Omega t$, and define $a \equiv \left(\frac{2\omega_o}{\Omega} \right)^2$ and $q \equiv \frac{2Y_o}{l}$, to give

$$\frac{4}{\Omega^2} \ddot{\psi} - (a - 2q \cos \Omega t) \sin \psi = 0 .$$

Finally, if we define $\tau = \frac{\Omega t}{2}$, then this can be slightly simplified to obtain:

$$\frac{d^2 \psi}{d\tau^2} - (a - 2q \cos \Omega t) \sin \psi = 0$$

which differs from Hand & Finch equation 10.36 only by a single $-$ sign. Note that *no* approximations were made in this derivation.

7. and 8. (double credit problem)

Hand & Finch, Problem 10.9 (a)-(d) *only* (how does a child pump a swing?)

Solution:

(a)

The Lagrangian for this system is

$$\mathcal{L} = \frac{m}{2} (\dot{l}^2 + l^2 \dot{\theta}^2) + mgl \cos \theta .$$

Apply the Euler-Lagrange equation:

$$\frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}} \right) - \frac{\partial \mathcal{L}}{\partial \theta} = 0$$

$$\frac{d}{dt} (ml^2 \dot{\theta}) + mgl \sin \theta = 0$$

If we let $\theta \ll 1$, then this becomes

$$\frac{d}{dt}(l^2\dot{\theta}) + gl\theta = 0 .$$

(b)

If we now substitute in $l(t) = \bar{l}(1 + \delta(t))$ and $\tau = \sqrt{\frac{g}{l}}t$, the above expression becomes:

$$\frac{d}{d\tau} \left((1 + \delta)^2 \dot{\theta} \right) + (1 + \delta)\theta = 0$$

Making the following substitutions

$$\theta = \frac{\phi}{l(t)} = \frac{1}{\bar{l}} \frac{\phi}{(1 + \delta)}$$

$$\dot{\theta} = \frac{1}{\bar{l}} \left(\frac{\dot{\phi}}{1 + \delta} - \frac{\phi\dot{\delta}}{(1 + \delta)^2} \right)$$

yields

$$\frac{1}{\bar{l}} \frac{d}{d\tau} \left((1 + \delta)^2 \left(\frac{\dot{\phi}}{1 + \delta} - \frac{\phi\dot{\delta}}{(1 + \delta)^2} \right) \right) + (1 + \delta) \frac{\phi}{\bar{l}(1 + \delta)} = 0$$

$$\frac{d}{d\tau} \left((1 + \delta)\dot{\phi} - \phi\dot{\delta} \right) + \phi = 0$$

$$\ddot{\phi} + \left(\frac{1 - \ddot{\delta}}{1 + \delta} \right) \phi = 0$$

(c)

For small δ ,

$$\left(\frac{1 - \ddot{\delta}}{1 + \delta} \right) \approx (1 - \ddot{\delta})(1 - \delta) \approx 1 - \delta - \ddot{\delta} ,$$

in which case the DE becomes

$$\ddot{\phi} + \phi = (\delta + \ddot{\delta})\phi .$$

Now, let

$$\delta = \delta_0 \cos 2\tau$$

$$\ddot{\delta} = -4\delta_0 \cos 2\tau$$

$$\phi = Al(0) \cos \tau + \mathcal{O}(\delta_0)$$

We obtain as an equation for ϕ

$$\ddot{\phi} + \phi = (\delta_0 \cos 2\tau - 4\delta_0 \cos 2\tau) Al(0) \cos \tau$$

$$= -3\delta_0 Al(0) \cos 2\tau \cos \tau$$

Using the trig identity $\cos \tau \cos 2\tau = \frac{1}{2}(\cos \tau + \cos 3\tau)$ gives (to first order in δ_0) the result:

$$\ddot{\phi} + \phi = -\frac{3}{2}\delta_0 Al(0) (\cos \tau + \cos 3\tau)$$

(d)

The homogenous solution of the above DE is of the form

$$\phi_h = B \sin \tau + C \cos \tau ,$$

where B and C will need to be chosen to satisfy the initial conditions (we will wait to do this until we have the particular solution as well). The particular solution to the above DE is the sum of the particular solutions ϕ_1 and ϕ_3 to the differential equations

$$\ddot{\phi}_1 + \phi_1 = -\frac{3}{2}\delta_0 Al(0) \cos \tau$$

$$\ddot{\phi}_3 + \phi_3 = -\frac{3}{2}\delta_0 Al(0) \cos 3\tau .$$

The equation for ϕ_3 is easier to solve, so let's do it first. Substitute $\phi_3 \equiv D \cos 3\tau$ to obtain

$$(-9D + D) \cos 3\tau = -\frac{3}{2}\delta_0 Al(0) \cos 3\tau$$

$$D = \frac{3}{16}\delta_0 Al(0) .$$

Turning to the equation for ϕ_1 , if we were to substitute $\phi_1 = E \cos \tau$ in analogy to the method we used for ϕ_3 , the LHS would vanish and the equation would not be satisfied. Instead (more or less by trial and error), we substitute $\phi_1 = F\tau \sin \tau$. (If there is a rationale, it is that the extra factor of τ can be expected to destroy the cancellation on the LHS, allowing some harmonic function of τ to survive.)

$$\left(\frac{d^2}{d\tau^2} + 1 \right) (F\tau \sin \tau) = -\frac{3}{2}\delta_0 Al(0) \cos \tau$$

$$\frac{d}{d\tau} (\sin \tau + \tau \cos \tau) + \tau \sin \tau = -\frac{3\delta_0 Al(0)}{2F} \cos \tau$$

$$2 \cos \tau = -\frac{3\delta_0 Al(0)}{2F} \cos \tau$$

$$F = -\frac{3}{4}\delta_0 Al(0) .$$

Putting it all together, the general solution for ϕ is

$$\begin{aligned}\phi &= \phi_h + \phi_1 + \phi_3 \\ &= B \sin \tau + C \cos \tau \\ &\quad + \delta_0 Al(0) \left(\frac{3}{16} \cos 3\tau - \frac{3}{4} \tau \sin \tau \right) .\end{aligned}$$

The variable ϕ is the product of two time-dependent functions:

$$\begin{aligned}\phi(\tau) &= \theta(\tau) l(\tau) \\ &= \theta(\tau) (\bar{l} + \delta_0 \cos 2\tau) .\end{aligned}$$

Taking its derivative with respect to τ ,

$$\dot{\phi}(\tau) = \dot{\theta}(\tau) (\bar{l} + \delta_0 \cos 2\tau) - 2\theta(\tau) \delta_0 \sin 2\tau .$$

Applying the initial conditions $\theta(0) = A$, $\dot{\theta}(0) = 0$, we obtain the initial conditions on ϕ :

$$\begin{aligned}\phi(0) &= A(\bar{l} + \delta_0) \\ &= Al(0) \\ \dot{\phi}(0) &= 0 .\end{aligned}$$

Finally we use these initial conditions to determine B and C :

$$\begin{aligned}Al(0) &= \phi(0) \\ &= C + \delta_0 Al(0) \left(\frac{3}{16} \right) \\ Al(0) \left(1 - \frac{3}{16} \delta_0 \right) &= C \\ 0 &= \dot{\phi}(0) \\ &= -B .\end{aligned}$$

Plugging these values for B and C into the general solution for ϕ , we obtain the complete expression for $\phi(\tau)$:

$$\begin{aligned}\phi(\tau) &= Al(0) \left(1 - \frac{3}{16} \delta_0 \right) \cos \tau \\ &\quad + \frac{3}{16} Al(0) \delta_0 \cos 3\tau - \frac{1}{4} Al(0) \delta_0 \tau \sin \tau .\end{aligned}$$

Clearly, the coefficient of the last term is increasing in magnitude linearly with time.